Combinatorial optimization and its applications in image Processing

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Part 1: Optimization in image processing

Optimization in image processing

Many image processing problems can be formulated as optimization problems - we define a function that assign a "goodness" value to every possible solution, and then seek a solution that is as "good" as possible.

- Image segmentation
- Image registrations/stereo matching/optical flow
- Image restoration/filtering

The "goodness" criterion is often referred to as an *objective function*.

Application 1: Image registration

Non-rigid Image registration

- Given two images, find a transformation (deformation field) that aligns one image to the other.
- Registration, stereo disparity, optical flow. . .

Example: Medical Image registration

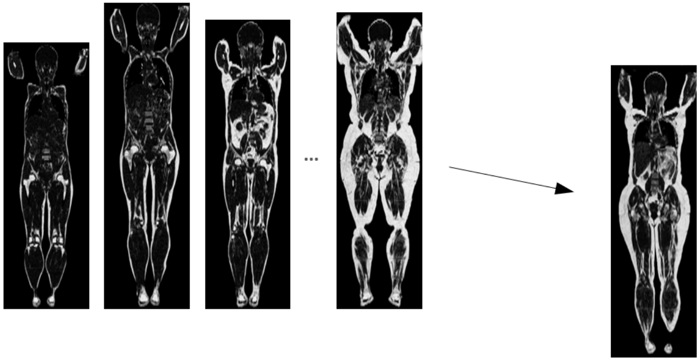


Figure 1: Registering a series of whole body MRI images to match a common "mean person" facilitates direct comparisons between subjects.

Example: Optical flow

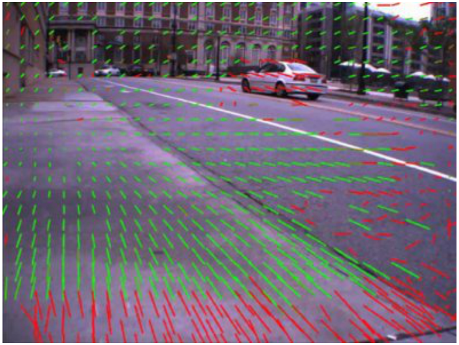


Figure 2: Registering consecutive frames in a video sequence facilitates, e.g., tracking and motion analysis.

Example: Optical flow for slowmotion

`https://www.youtube.com/watch?v=xLqz09-3vHg`

Figure 3: By computing optical flow in a video sequence, it is possible to interpolate new frames inbetween the captured ones, to simulte slow motion photography.

Image registration as optimization

- Image registration can be formulated as an optimization problem.
- Typically, we seek a solution that maximizes some notion of similarity between the images, while also maintaining some degree of smoothness of the deformation field.

Application 2: Image segmentation

Image segmentation

- Image segmentation is the task of partitioning an image into relevant objects and structures.
- Image segmentation is an ill-posed problem...

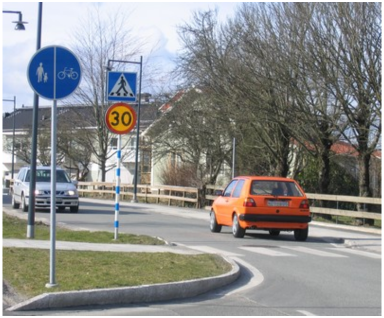


Figure 4: What do we mean by a segmentation of this image?

Image segmentation

- Image segmentation is the task of partitioning an image into relevant objects and structures.
- Image segmentation is an ill-posed problem...
- ...Unless we specify a segmentation *target*.

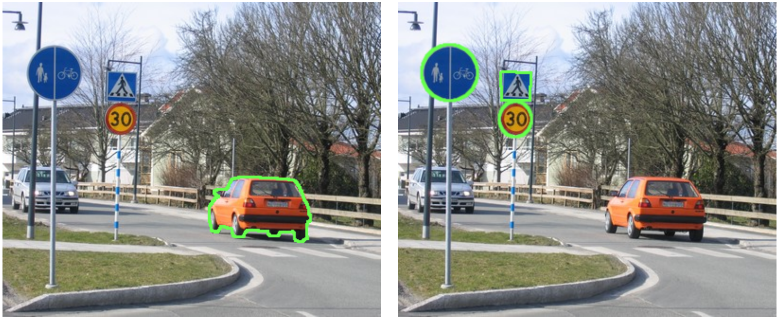


Figure 5: Segmentation relative to semantically defined targets.

Image segmentation

We can divide the image segmentation problem into to two tasks:

- *Recognition* is the task of roughly determining where in the image an object is located.
- *Delineation* is the task of determining the exact extent of the object.

Image segmentation

- *Recognition* information can be provided, e.g, by interactive annotations from a human user, or by an automatic algorithm incorporating high level knowledge.
- *Delineation* information can often be extracted from low-level image features. Typical such terms may favour:
  - Segmentations where object boundaries coincide with strong edges in the image.
  - Segmentations that divide the image into regions that are homogeneous with respect to some feature (intensity, color, texture).

Paradigms for user input: Seeded segmentation

- The user is asked to provide correct segmentation labels for a subset of the image elements ("seed-points")

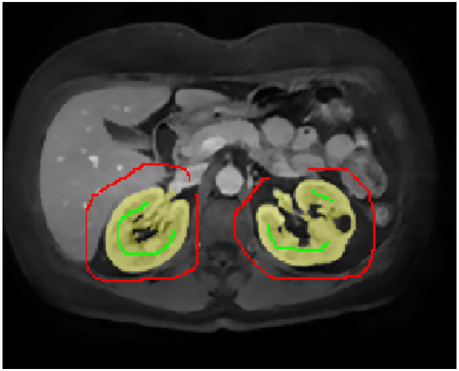


Figure 6: Segmentation with regional constraints.

Segmentation as an optimization problem

- We wish to find a segmentation that is *as good as possible* according to some criterion, based on recognition and delineation terms.

Application 3: Image restoration/filtering

Image filtering as an optimization problem

- Some image filtering operations can be formulated as an optimization problem with objective functions balancing two criteria:
  - The filtered image should be similar to the original one.
  - The filtered image should be smooth. (e.g. have small gradients)
- A Gaussian filter, for example, can be viewed as the solution to such an optimization problem.

Image filtering as an optimization problem, why?

- What do we gain from viewing filtering as an optimization problem?
- Perhaps not that much, for ordinary filtering operations such as Gaussian filters.
- But it can be useful to keep this view if we want to develop new filters, e.g., edge preserving *anisotropic* filters ...

Example: Edge preserving filter

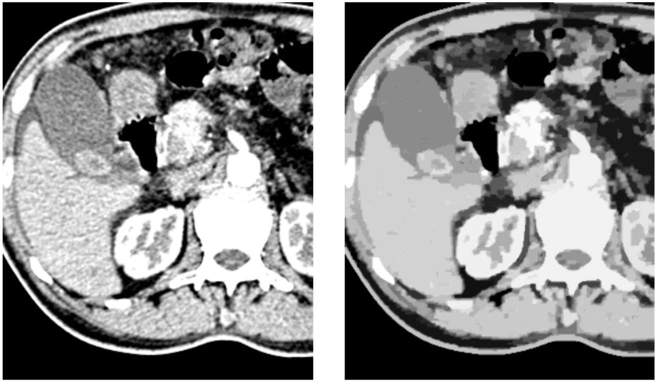


Figure 7: Image by Couprie et al.

"Typical" optimization problems in image analysis

The optimization problems occuring in the applications studied so far have a number of things in common:

- They are *pixel labeling* problems. In all cases, we seek to assign some type of *labels* (values) to the pixels of the image:
    - Object classes for segmentation.
    - Displacement vectors for registration.
    - Intensities/colours for restoration.
- The objective function consists of two terms:
    - A *data term* that measures how appropriate a label is for a certain pixel given some prior knowledge.
    - A *smoothness term* that favours spatial coherency.

Throughout the course, we will study optimization problems of this type.

Part 2: Combinatorial optimization

Combinatorial optimization

- A combinatorial optimization problem consists of a *finite* set of candidate solutions $\mathcal{S}$ and an objective function $f : \mathcal{S} \to \mathbb{R}$.
- In our examples, $\mathcal{S}$ will be typically be the set of all maps from the vertices of a graph to some set of *labels*.
- The objective function function f can measure either "goodness" or "badness" of a solution. Here, we assume that we want to find a solution $x \in \mathcal{S}$ that minimizes f.
- Ideally, we want to find a *globally minimal* solution, i.e., a solution $x^* \in \operatorname*{argmin}_{x \in \mathcal{S}} f(x)$.

Combinatorial optimization

- It is tempting to view the objective function and the optimization method as completely independent. This would allow us to design an objective function (and a solution space) that describes the problem at hand, and apply general purpose optimization techniques.
- For an arbitrary objective function, finding a global optima recuires checking all solutions.
- The set $\mathcal{S}$ of solutions is finite. Can't we just search this set for the globally optimal solution?

How hard is combinatorial optimization?

- In vertex labeling, the number of possible solutions is $|L|^{|V|}$.
- Consider binary labeling of a 256×256 image.
- The number of possible solutions is 2^{65536}. This is a ridiculously large number!
- Searching the entire solution space for a global optimum is not feasible!

So, what do we do?

- For restricted classes of optimization problems, it is sometimes possible to design efficient algorithms that are guaranteed to find global optima. In upcoming lectures, we will cover some of these.
- Local search methods can be used to find *locally optimal solutions*. This is the topic of the remainder of this lecture.

Local optimality

- Define a neighborhood system $\mathcal{N}$ that specifies, for any candidate solution x, a set of *nearby* candidates $\mathcal{N}(x)$.
- A *local minimum* is a candidate x^* such that $f(x^*) \leq \min_{x \in \mathcal{N}(x^*)} f(x)$.

Local search

- A general method for finding local minima.
  - Start at an arbitrary solution.
  - While the current solution is not a local minimum, replace it with an adjacent solution for which f is lower.
- This algorithm is guaranteed to find a locally optimal solution in a finite number of iterations. (Proving this statement is one of the exercises!)

Local search spaces as graphs

- We have a set $\mathcal{S}$ and an adjacency relation N.
- It's a (huge) graph!
- We never store this graph explicitly, but it can be useful to consider.
- For example, it seems reasonable to define the adjacency relation so that the graph of the search space is connected.

Local search

- "This algorithm is guaranteed to find a locally optimal solution in a finite number of iterations. Why?"
  - The number of solutions is finite.
  - *If* the algorithm terminates, the result is a local minimum. (Why?)
  - Each connected component in the graph of the search space contains at least one local minimum. (Why?)
  - A solution is never visited more than once. (Why?)

Best-improvement search

- In *best-improvement search*, we consider *all* states in the local neighborhood of the current state. We accept the one that best improves the objective function.
- In *first-improvement search*, we consider the states in the local neighborhood of the current state one at a time. We accept the first one that improves upon the current state.
- Which one gives the best results? Which leads to a faster algorithm?

Local search with restarts

- Run the algorithm several times, with different initialization.
- "Patience" factor. (Terminate local search if no local optimum find within k steps)
- With infinite patience, we will find a locally optimal solution with probability 1.
- With infinite restarts, we will find a globally optimal solution with probability 1.

Local search, an example

Let's take a look simple binary thresholding

- Let $I(v)$ be the intensity of the pixel corresponding to v.
- Given a threshold t, we compute a vertex labeling according to:

$$\mathcal{L}(v) = \begin{cases} \textit{foreground} & \text{if } I(v) \geq t \\ \textit{background} & \text{otherwise} \end{cases} . \tag{1}$$

- Next, we will reformulate this as an optimization problem.

Local search, an example

We define the objective function f as

$$f = \sum_{v \in V} \Phi(v) \ , \tag{2}$$

where

$$\Phi(v) = \begin{cases} abs(max(t - I(v), 0)) & \text{if } \mathcal{L}(v) = foreground \\ abs(max(I(v) - t, 0)) & \text{otherwise} \end{cases} \quad . \tag{3}$$

Local search, example

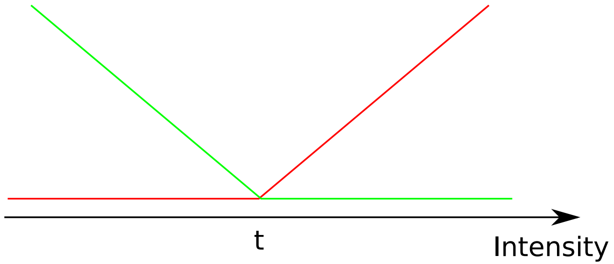


Figure 8: Objective function for binary thresholding. The red curve is the cost of assigning the label “background” to a vertex with a certain intensity, and the green curve is the cost of assigning the “foreground” label.

Optimization by local search

- We say that two vertex labelings are adjacent if we can turn one into the other by changing the label of *one* vertex.
- We start from an arbitrary labeling, and use first-improvement search to find a locally optimal solution.

Optimization by local search, algorithm

```
done=false
while done do
    done=true
    foreach pixel p in the image do
        Can we improve the current solution by changing the label of p?
        If so, change the label and set done=false.
    end
end
```

Local search, an example

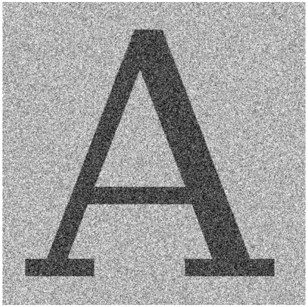


Figure 9: Thresholding as an optimization problem.

Local search, an example

- Start from an arbitrary labeling.
- In this case, the label of each pixel does not depend on the label of any other pixels, so a local optimum is reached after only one iteration of the while-loop.
- (This optimum is in fact also global)

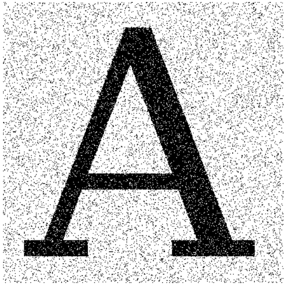


Figure 10: Thresholding as an optimization problem.

Local search, an example

- Let us add to the objective function a smoothness term $|\partial\mathcal{L}|$, that penalizes long boundaries:

$$f = \sum_{v \in V} \Phi(v) + \alpha|\partial\mathcal{L}| \ , \qquad (4)$$

where α is a real number that controls the degree of "smoothing".

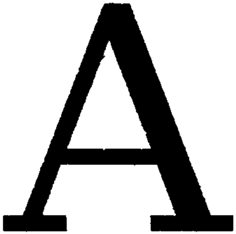


Figure 11: Thresholding with smoothness term.

Unary and binary terms

- The data term ϕ in the example is a sum over the pixels in the image. In this term, each pixel is considered individually. We say that ϕ is a *unary* term.
- In contrast, the smoothness term is defined over all *pairs* of adjacent pixels (edges, in the graph context). We say that this term is *binary*.

Local search, an example

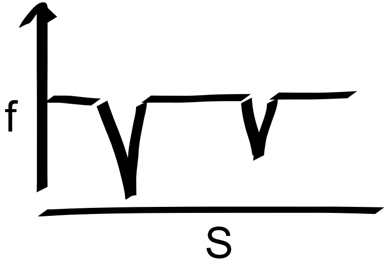
- After adding the (binary) smoothness term, we have introduced a dependency between the labels of adjacent pixels. We can no longer decide on the best label for each pixel independently!
- This makes the optimization problem harder to solve.
    - The local search algorithm requires many passes over the image before convergence.
    - The local solution is no longer guaranteed to be a global optimum.

A note on efficient implementation

- In our example, the objective is a sum over all pixels in the image (and all edges in the cut corresponding to the current segmentation).
- Evaluating the entire objective function at each iteration is expensive.
- Instead, we can calculate how much the objective function *changes* when we change the label of a vertex.
- This is good to keep in mind when designing the objective function.

When is local search useful?

Similar solutions should have similar costs ("continuous" objective function).



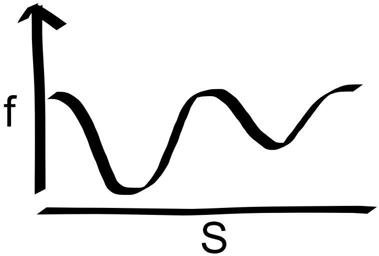


Figure 12: (Left) An objective function that is hard to optimize using local search (Right) An objective function that is possible to optimize using local search.

Very large-scale neighborhood search

- To avoid getting trapped in poor local minima, it is desirable to use as large neighborhoods as possible.
- ...but large neighborhoods lead to slow computations.
- For some problems, we can find efficient algorithms for computing globally optimal solution within a subset of $\mathcal{S}$. If we use this subset as our local neighborhood, we can do best-improvement search!
- We will look at one such technique in an upcoming lecture.

Global optimization

- For a general combinatorial optimization problem, finding the optimal solution requires checking all solutions.
- For specific classes of problems, we can do better!
- Quite remarkably, there are many algorithms for solving optimization problems of interest in image analysis that guarantee globally optimal results
- In this course we will cover some of the most important such methods.

Summary

- Many image analysis problems can be formulated as (combinatorial) optimization problems.
- Local search methods can be used to find *locally* optimal solutions to any combinatorial optimization problem.
- Depending on the problem and the local search strategy used, these locally optimal solutions may or may not be good enough.
- For many interesting combinatorial optimization problems we can find globally optimal solutions efficiently. More on this later!